

# Global Lipschitz Stability in Determining Coefficients of the Radiative Transport Equation

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## Abstract

In this article, for a radiative transport equation, we study inverse problems of determining a time independent scattering coefficient or a total attenuation by boundary data on the complementary sub-boundary after making one time input of a pair of a positive initial value and boundary data on a suitable sub-boundary. The main results are Lipschitz stability estimates. We can also prove the reverse inequality, which means that our estimates for the inverse problems are the best possible. The proof is based on a Carleman estimate.

## 1 Radiative Transport Equation and Main Results

We consider wave or particles propagating in a random medium. Let  $\Omega$  be a bounded domain of  $\mathbb{R}^d$ ,  $d \geq 2$  with  $C^1$ -boundary  $\partial\Omega$ . The scalar product in  $\mathbb{R}^d$  is denoted by a dot  $(\cdot)$ . Let  $\nabla = \nabla_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}\right)$ . We let  $u(x, v, t) \in \mathbb{R}$  denote the angular

density at time  $t > 0$  and position  $x \in \mathbb{R}^d$  with the velocity  $v \in V$ , where  $V = \{v \in \mathbb{R}^d; 0 < v_0 \leq |v| \leq v_1\}$ .

Let  $\sigma_a(x, v)$  and  $\sigma_s(x, v)$  denote the absorption and scattering coefficients, respectively. Note that  $\sigma_a$  and  $\sigma_s$  are positive measurable functions:

$$\sigma_a : \Omega \times V \rightarrow \mathbb{R}, \quad \sigma_s : \Omega \times V \rightarrow \mathbb{R}. \quad (1.1)$$

We introduce the total attenuation as  $\sigma_t = \sigma_a + \sigma_s$ . The following radiative transport equation, which is a linearized Boltzmann equation, governs  $u(x, v, t)$  for  $x \in \Omega$ ,  $v \in V$ ,  $0 < t < T$ ,

$$Pu := P_0 u(x, v, t) + \sigma_t(x, v)u - \sigma_s(x, v) \int_V p(x, v, v') u(x, v', t) dv' = 0, \quad (1.2)$$

where

$$P_0 u := \partial_t u(x, v, t) + v \cdot \nabla u(x, v, t). \quad (1.3)$$

The phase function  $p(x, v, v')$  satisfies

$$\int_V p(x, v, v') dv' = 1 \quad \text{for all } (x, v). \quad (1.4)$$

Equation (1.2) describes transport in a random medium such as light in biological tissue [1, 2], neutrons in a reactor [8, 11], and light in the interstellar medium [9] and atmospheres [29]. We let  $\nu(x)$  be the outward normal unit vector to  $\partial\Omega$  at  $x \in \partial\Omega$ . We define  $\Gamma_{\pm}$  as

$$\Gamma_+ = \{(x, v) \in \partial\Omega \times V; \nu(x) \cdot v > 0\}, \quad \Gamma_- = \{(x, v) \in \partial\Omega \times V; \nu(x) \cdot v < 0\}. \quad (1.5)$$

We impose the following boundary conditions.

$$u(x, v, 0) = a(x, v), \quad x \in \Omega, \quad v \in V, \quad (1.6)$$

$$u(x, v, t) = g(x, v, t), \quad 0 < t < T, \quad (x, v) \in \Gamma_-. \quad (1.7)$$

We consider inverse problems of determining  $\sigma_t$  or  $\sigma_s$  by boundary data  $u(x, v, t)$ ,  $(x, v) \in \Gamma_+$ ,  $0 < t < T$  after setting up an initial value (1.6) and boundary value (1.7) once. Our inverse problem is motivated by optical tomography, in which we recover  $\sigma_t$  and  $\sigma_s$  from boundary measurements (e.g., [1, 2]). An incident laser beam  $g(x, v, t)$  enters the sample on the boundary  $\Gamma_-$ , and the outgoing light  $u(x, v, t)$  is measured on the boundary  $\Gamma_+ \times (0, T)$ .

We refer to works concerning inverse problems on transport equation. Choulli and Stefanov [10] proved the uniqueness of  $\sigma_t$  and  $\sigma_s$  for the scattering operator. We define the albedo operator  $\mathcal{A}$  as

$$\mathcal{A}[g] = u(x, v, t), \quad (x, v) \in \Gamma_+, \quad 0 < t < T, \quad (1.8)$$

by assuming that the initial value is zero. Stability in determining some coefficients among  $\sigma_t$ ,  $\sigma_s$ ,  $p$  is proved by the angularly averaged albedo operator [4] and by the full albedo operator [5]. For the inverse problems in Refs. [4] and [5], the input-output operation can be limited to the boundary and the initial value can be zero, but one has to make infinitely many measurements. For the stationary transport equation, the non-uniqueness in the coefficient inverse problem with the albedo operator was characterized by gauge equivalent pairs in [31], and the Lipschitz stability for gauge equivalent classes was proved for the time-independent radiative transport equation in [27]. See also review articles [3, 30] for coefficient inverse problems for the radiative transport equation.

Klibanov and Pamyatnykh [23] proved the uniqueness of  $\sigma_t$  by the boundary values of  $u$ . The formulation in [23] is different from [4], [5], [10] and measures a single output on  $\Gamma_+ \times (0, T)$  after choosing initial value and boundary data on  $\Gamma_- \times (0, T)$ .

In this article, we adopt the same formulation as in [23] and we consider the inverse problems of determining  $\sigma_s$  or  $\sigma_t$  by boundary value on  $\Gamma_+ \times (0, T)$  with a suitable single input of the initial value. Our main results are Lipschitz stability estimates in determining  $\sigma_s$  or  $\sigma_t$ . To the best knowledge of the authors, there are no publications on the Lipschitz stability with a single measurement data. Bukhgeim and Klibanov [6] proposed a methodology for proving the uniqueness and the stability for inverse problems with a single measurement, on which [23] is based. Their method uses an  $L^2$ -weighted estimate called a Carleman estimate for solutions to the differential equation under consideration. The Carleman estimate dates back to Carleman [7]. See Hörmander [13], Isakov [18], and Lavrent'ev, Romanov, and Shishat'skii [26]. As for inverse problems by Carleman estimate, we refer for example to Imanuvilov and Yamamoto [15], [16], Isakov [17], [19], Klibanov [20], [21], Klibanov and Timonov [24], and Yamamoto [32]. Moreover see Klibanov and Pamyatnykh [22] for the Carleman estimate for a transport equation and an application to the unique continuation, and Klibanov and Yamamoto [25] for the exact controllability for the transport equation. Prilepkov and Ivankov [28] discusses an inverse problem of determining a  $t$ -function in the case where  $\sigma_t$  depends on  $x, v, t$ .

Throughout this article,  $H^m(\Omega)$  denotes usual Sobolev spaces. We set

$$X = H^1(0, T; L^\infty(\Omega \times V)) \cap H^2(0, T; L^2(\Omega \times V)).$$

For arbitrarily fixed constant  $M > 0$ , we set

$$\mathcal{U} = \{u \in X; \|u\|_X + \|\nabla u\|_{H^1(0, T; L^2(\Omega \times V))} \leq M\}. \quad (1.9)$$

Now we are ready to state our main results.

**Theorem 1 (determination of  $\sigma_t$ )**

Let  $u^k = u(\sigma_t^k)(x, v, t)$ ,  $k = 1, 2$  be solutions to the transport equation:

$$\begin{aligned} \partial_t u(x, v, t) + v \cdot \nabla u + \sigma_t^k(x, v)u - \sigma_s(x, v) \int_V p(x, v, v')u(x, v', t)dv' &= 0, \\ u(x, v, 0) &= a(x, v), \quad x \in \Omega, v \in V, k = 1, 2, \\ u &= g \quad \text{on } \Gamma_- \times (0, T). \end{aligned}$$

Let  $u^k \in \mathcal{U}$  and  $\|\sigma_t^k\|_{L^\infty(\Omega \times V)}, \|\sigma_s\|_{L^\infty(\Omega \times V)} \leq M$ . We assume that

$$T > \frac{1}{v_0} \sup_{x \in \Omega} |x - x_0| \quad (1.10)$$

with some  $x_0 \notin \overline{\Omega}$  and

$$a(x, v) > 0, \quad (x, v) \in \overline{\Omega \times V}. \quad (1.11)$$

Then there exists a constant  $C = C(M) > 0$  such that

$$\begin{aligned} C^{-1} \left( \int_0^T \int_{\Gamma_+} (\nu(x) \cdot v) |\partial_t(u^1 - u^2)(x, v, t)|^2 dS dv dt \right)^{\frac{1}{2}} &\leq \|\sigma_t^1 - \sigma_t^2\|_{L^2(\Omega \times V)} \\ &\leq C \left( \int_0^T \int_{\Gamma_+} (\nu(x) \cdot v) |\partial_t(u^1 - u^2)(x, v, t)|^2 dS dv dt \right)^{\frac{1}{2}}. \end{aligned} \quad (1.12)$$

**Theorem 2 (determination of  $\sigma_s$ )**

Let  $u^k = u(\sigma_s^k)(x, v, t)$ ,  $k = 1, 2$ , be the solution to the transport equation:

$$\begin{aligned} \partial_t u(x, v, t) + v \cdot \nabla u + \sigma_t(x, v)u - \sigma_s^k(x, v) \int_V p(x, v, v')u(x, v', t)dv' &= 0, \\ u(x, v, 0) &= a(x, v), \quad x \in \Omega, v \in V, \\ u &= g \quad \text{on } \Gamma_- \times (0, T), \quad k = 1, 2. \end{aligned}$$

Let  $u^k \in \mathcal{U}$  and  $\|\sigma_t\|_{L^\infty(\Omega \times V)}, \|\sigma_s^k\|_{L^\infty(\Omega \times V)}, k = 1, 2$ . We assume (1.10) and (1.11).

Then there exists a constant  $C = C(M) > 0$  such that

$$C^{-1} \left( \int_0^T \int_{\Gamma_+} (\nu(x) \cdot v) |\partial_t(u^1 - u^2)(x, v, t)|^2 dS dv dt \right)^{\frac{1}{2}} \leq \|\sigma_s^1 - \sigma_s^2\|_{L^2(\Omega \times V)}$$

$$\leq C \left( \int_0^T \int_{\Gamma_+} (\nu(x) \cdot v) |\partial_t(u^1 - u^2)(x, v, t)|^2 dS dv dt \right)^{\frac{1}{2}}. \quad (1.13)$$

In (1.12) and (1.13), the second inequalities show the Lipschitz stability for the inverse problems, while the first inequalities are related to the initial/boundary value problems in which we are required to find  $\partial_t u$  on  $\Gamma_+ \times (0, T)$  for given  $a$  and  $(\sigma_t^k, \sigma_s)$ ,  $(\sigma_t, \sigma_s^k)$ ,  $k = 1, 2$ . We obtain both-sided estimates and so the estimates for the inverse problems are the best possible.

For the Lipschitz stability for the inverse problems, we need the positivity (1.11) up to the boundary  $\partial(\Omega \times V)$  of the initial value. Measurements must be set up so that this positivity is guaranteed. The positivity condition is restricting but can be achieved in practice for example as follows. Let us consider optical tomography of the human brain (cf. [12, 14]). We use a continuous-wave near-infrared laser beam and modulate the light by using an optical device. Before being temporally varied, the time-independent light is applied to the head. The light is then scattered in different directions in the brain, and comes out. Thus, in this setup, we can consider that the initial angular density  $a(x, v)$  in the head is positive in  $\overline{\Omega \times V}$ .

Moreover we have to assume (1.10), that is, the observation time  $T$  should be large compared to the size of the domain  $\Omega$ . This is a natural condition because the transport equation has a finite propagation speed, which can be seen by (1.3).

In order to prove Theorems 1 and 2, it is sufficient to prove the linearized inverse problem. More precisely,

### **Theorem 3**

*We consider*

$$\partial_t u + v \cdot \nabla u + \sigma_t u - \sigma_s \int_V p(x, v, v') u(x, v', t) dv' = f(x, v) R(x, v, t), \quad x \in \Omega, v \in V, 0 < t < T,$$

$$u(x, v, 0) = 0, \quad x \in \Omega, v \in V,$$

$$u = 0 \quad \text{on } \Gamma_- \times (0, T).$$

*We assume*

$$R, \partial_t R \in L^2(0, T; L^\infty(\Omega \times V)), \quad \sigma_t, \sigma_s \in L^\infty(\Omega \times V),$$

*and*

$$u \in H^2(0, T; L^2(\Omega \times V)) \cap H^1(0, T; L^2(\Omega \times V)), \quad \nabla u \in H^1(0, T; L^2(\Omega \times V)).$$

*We further assume (1.10) and (1.11). Moreover let*

$$R(x, v, 0) > 0, \quad (x, v) \in \overline{\Omega \times V}.$$

Then there exists a constant  $C > 0$  such that

$$C^{-1} \left( \int_0^T \int_{\Gamma_+} (\nu \cdot v) |\partial_t u|^2 dS dv dt \right)^{\frac{1}{2}} \leq \|f\|_{L^2(\Omega \times V)} \leq C \left( \int_0^T \int_{\Gamma_+} (\nu \cdot v) |\partial_t u|^2 dS dv dt \right)^{\frac{1}{2}} \quad (1.14)$$

for any  $f \in L^2(\Omega \times V)$ .

In fact, for the proof of Theorem 1, setting  $u = u^1 - u^2$ ,  $f = \sigma_t^1 - \sigma_t^2$  and  $R = -u^2$ , we have the above linearized inverse problem. By the regularity assumption of  $u^1, u^2$ , we can apply Theorem 3 to obtain the conclusion (1.12). We can similarly derive Theorem 2 from Theorem 3.

The article is composed of 4 sections. In section 2, we prove the first inequality of (1.14). In section 3, we prove a key Carleman estimate and in section 4, we complete the proof of Theorem 3.

## 2 Proof of the first inequality in Theorem 3

We set

$$Q = \Omega \times V.$$

In this section,  $C > 0$  denotes generic constants which are independent of  $f$ .

We prove the first inequality by energy estimation. We set

$$u_1 = \partial_t u.$$

Then

$$Pu_1(x, v, \xi) = f(x, v) \partial_t R(x, v, \xi), \quad (x, v) \in Q, \quad 0 < \xi < t.$$

Multiplying  $Pu_1(x, v, \xi) = f \partial_t R$  by  $2u_1(x, v, \xi)$  and integrating over  $(x, v, \xi) \in Q \times (0, t)$ , we have

$$\begin{aligned} & \int_0^t \partial_t \left( \int_Q |\partial_t u(x, v, \xi)|^2 dx dv \right) d\xi + \int_0^t \int_Q v \cdot \nabla (|\partial_t u(x, v, \xi)|^2) dx dv d\xi + \int_0^t \int_Q 2\sigma_t |\partial_t u|^2 dx dv d\xi \\ &= \int_0^t \left( \int_Q 2\sigma_s \partial_t u(x, v, \xi) \left( \int_V p(x, v, v') \partial_t u(x, v', \xi) dv' \right) dx dv \right) d\xi \\ &+ \int_0^t \int_Q 2(\partial_t u)(x, v, \xi) f(x, v) (\partial_t R)(x, v, \xi) dx dv d\xi. \end{aligned} \quad (2.1)$$

Integrating by parts and noting that  $\sigma_t \in L^\infty(Q)$ , for  $0 \leq t \leq T$ , we have

$$\begin{aligned} & \int_0^t \int_Q v \cdot \nabla (|\partial_t u|^2) dx dv d\xi = \int_0^t \int_V \int_{\partial\Omega} (\nu \cdot v) |\partial_t u|^2 dS dv d\xi \\ = & \int_0^t \left( \int_{\Gamma_+} + \int_{\Gamma_-} \right) (\nu \cdot v) |\partial_t u|^2 dS dv d\xi \geq \int_0^t \int_{\Gamma_+} (\nu \cdot v) |\partial_t u(x, v, \xi)|^2 dS dv d\xi \end{aligned}$$

and

$$\int_0^t \int_Q 2|\sigma_t| |\partial_t u|^2 dx dv d\xi \leq C \int_0^t \int_Q |\partial_t u(x, v, \xi)|^2 dx dv d\xi.$$

Since  $u(x, v, \xi) = \int_0^\xi \partial_t u(x, v, \eta) d\eta$  by  $u(x, v, 0) = 0$ , we have

$$\begin{aligned} & |[\text{the first term on right-hand side of (2.1)}]| \\ \leq & C \int_0^t \left( \int_Q |\partial_t u(x, v, \xi)| \left( \int_V |\partial_t u(x, v', \xi)| dv' \right) dx dv \right) d\xi \\ = & C \int_0^t \left( \int_Q |\partial_t u(x, v, \xi)| \left( \int_V \left| \int_0^\xi \partial_t u(x, v', \eta) d\eta \right| dv' \right) dx dv \right) d\xi \\ \leq & C \int_0^t \int_Q |\partial_t u(x, v, \xi)| \left( \int_V \int_0^\xi |\partial_t u(x, v', \eta)| d\eta dv' \right) dx dv d\xi \\ \leq & C \int_0^t \int_Q \left( \int_V \int_0^\xi (|\partial_t u(x, v, \xi)|^2 + |\partial_t u(x, v', \eta)|^2) d\eta dv' \right) dx dv d\xi \\ \leq & C \int_0^t \int_Q |\partial_t u(x, v, \xi)|^2 dx dv d\xi, \quad 0 \leq t \leq T. \end{aligned}$$

At the second last inequality, we used

$$|\partial_t u(x, v, \xi)| |\partial_t u(x, v', \eta)| \leq |\partial_t u(x, v, \xi)|^2 + |\partial_t u(x, v', \eta)|^2.$$

Therefore, since

$$\int_0^t \partial_t \left( \int_Q |\partial_t u(x, v, \xi)|^2 dx dv \right) d\xi = \int_Q |\partial_t u(x, v, t)|^2 dx dv - \int_Q |f(x, v) R(x, v, 0)|^2 dx dv$$

by  $u(x, v, 0) = 0$ , applying the Cauchy-Schwarz inequality to the second term on the right-hand side of (2.1), we obtain

$$\begin{aligned} & \int_Q |\partial_t u(x, v, t)|^2 dx dv \\ \leq & \int_Q |\partial_t u(x, v, t)|^2 dx dv + \int_0^t \int_{\Gamma_+} (\nu \cdot v) |\partial_t u(x, v, \xi)|^2 dS dv d\xi \end{aligned} \tag{2.2}$$

$$\begin{aligned}
&\leq C \int_0^t \int_Q |\partial_t u(x, v, \xi)|^2 dx dv d\xi + C \int_0^T \int_Q |f|^2 |\partial_t R|^2 dx dv d\xi \\
&+ \int_Q |f(x, v) R(x, v, 0)|^2 dx dv, \quad 0 \leq t \leq T.
\end{aligned}$$

By  $R(\cdot, \cdot, 0) \in L^\infty(Q)$  and  $\partial_t R \in L^2(0, T; L^\infty(Q))$ , applying the Gronwall inequality to (2.2), we obtain

$$\int_Q |\partial_t u(x, v, t)|^2 dx dv \leq C \|f\|_{L^2(Q)}^2, \quad 0 \leq t \leq T. \quad (2.3)$$

Applying this to the right-hand side of (2.2), we obtain

$$\int_0^t \int_{\Gamma_+} (\nu(x) \cdot v) |\partial_t u(x, v, \xi)|^2 dS dv d\xi \leq C \|f\|_{L^2(Q)}^2, \quad 0 \leq t \leq T.$$

Thus the proof of the first inequality is completed.

### 3 Key Carleman estimate

We set  $y = y(x, t)$ ,  $u = u(x, v, t)$  and

$$Ly(x, t) = \partial_t y + v \cdot \nabla y, \quad x \in \Omega, t > 0, \quad (3.1)$$

where  $v \in V$  is arbitrarily fixed. We define

$$\varphi(x, t) = |x - x_0|^2 - \beta t^2 \quad (3.2)$$

where  $0 < \beta < v_0^2$ . Then we can prove the key Carleman estimate.

#### Lemma 3.1

(i) *There exist constants  $s_0 > 0$  and  $C > 0$  such that*

$$\begin{aligned}
&\int_{-T}^T \int_\Omega s |y(x, t)|^2 e^{2s\varphi} dx dt \\
&\leq C \int_{-T}^T \int_\Omega |Ly|^2 e^{2s\varphi(x, t)} dx dt + Cs \int_{-T}^T \int_{v \cdot \nu(x) \geq 0} (\nu \cdot v) y^2 e^{2s\varphi(x, t)} dS dt
\end{aligned}$$

for all  $s \geq s_0$  and  $y \in L^2(-T, T; H^1(\Omega)) \cap H_0^1(-T, T; L^2(\Omega))$  such that  $y(x, t) = 0$ ,  $0 < t < T$  if  $x \in \partial\Omega$  satisfies  $\nu(x) \cdot v \leq 0$ .

(ii) *There exist constants  $s_0 > 0$  and  $C > 0$  such that*

$$\int_{-T}^T \int_Q s |u(x, v, t)|^2 e^{2s\varphi} dx dv dt$$



$$\leq C \int_{-T}^T \int_Q |Lu|^2 e^{2s\varphi(x,t)} dx dv dt + Cs \int_{-T}^T \int_{\Gamma_+} (\nu \cdot v) u^2 e^{2s\varphi(x,t)} dS dv dt$$

for all  $s \geq s_0$  and  $y \in L^2(-T, T; L^2(Q)) \cap H_0^1(-T, T; L^2(Q))$  such that  $\nabla y \in L^2(-T, T; L^2(Q))$  and  $y = 0$  on  $\Gamma_- \times (-T, T)$ .

**Proof** The part (ii) is proved by integrating the conclusion of (i) and so it suffices to prove (i). We set  $z(x, t) = e^{s\varphi(x,t)} y(x, t)$  and  $(L_s z)(x, t) = e^{s\varphi(x,t)} L(e^{-s\varphi} z)$ . Then

$$L_s z = \partial_t z + v \cdot \nabla z - s((\partial_t \varphi) + (v \cdot \nabla \varphi))z := \partial_t z + v \cdot \nabla z - sA(x, t)z.$$

Here we set  $A = \partial_t \varphi + (v \cdot \nabla \varphi) = -2\beta t + 2v \cdot (x - x_0)$ . Hence by  $y|_{\Gamma_-} = 0$ , we have

$$\begin{aligned} & \int_{-T}^T \int_{\Omega} |Ly|^2 e^{2s\varphi(x,t)} dx dt = \int_{-T}^T \int_{\Omega} |L_s z|^2 dx dt \\ \geq & -2s \int_{-T}^T \int_{\Omega} A(\partial_t z + v \cdot \nabla z) z dx dt = -s \int_{-T}^T \int_{\Omega} (A \partial_t(z^2) + Av \cdot \nabla(z^2)) dx dt \\ = & s \int_{-T}^T \int_{\Omega} (\partial_t A + \nabla A \cdot v) z^2 dx dt - s \int_{-T}^T \int_{\partial\Omega} A(\nu \cdot v) z^2 dS dt \\ = & 2s(|v|^2 - \beta) \int_{-T}^T \int_{\Omega} z^2 dx dt - s \int_{-T}^T \int_{\partial\Omega} A(\nu \cdot v) z^2 dS dt \\ \geq & 2s(v_0^2 - \beta) \int_{-T}^T \int_{\Omega} z^2 dx dt - Cs \int_{-T}^T \int_{(\nu(x) \cdot v) \geq 0} (\nu \cdot v) z^2 dS dt \end{aligned}$$

Substituting  $z = e^{s\varphi} y$ , we complete the proof.

## 4 Proof of Theorem 3

The proof is similar to Imanuvilov and Yamamoto [15], [16]. We set

$$u_1 = \partial_t u.$$

Then we have

$$\begin{aligned} Pu_1 &= f(x, v) \partial_t R(x, v, t), \quad (x, v) \in Q, \quad 0 < t < T, \\ u_1(x, v, 0) &= f(x, v) R(x, v, 0), \quad (x, v) \in Q. \end{aligned}$$

We extend  $y, \sigma_s, \sigma_t, p$  as follows.

$$y(x, v, t) = \begin{cases} \partial_t u(x, v, t), & t > 0, \\ \partial_t u(x, -v, -t), & t < 0, \end{cases}$$

$$\sigma_s(x, v, t) = \begin{cases} \sigma_s(x, v), & t > 0, \\ -\sigma_s(x, -v), & t < 0, \end{cases} \quad \sigma_t(x, v, t) = \begin{cases} \sigma_t(x, v), & t > 0, \\ -\sigma_t(x, -v), & t < 0, \end{cases}$$

and

$$p(x, v, v', t) = \begin{cases} p(x, v, v'), & t > 0, \\ -p(x, -v, v'), & t < 0. \end{cases}$$

We set  $\tilde{R} = \partial_t R$ , and we extend  $\tilde{R}$  by  $\tilde{R}(x, v, t) = -\tilde{R}(x, -v, -t)$  for  $t < 0$ . Therefore, for  $t < 0$ , we have  $\partial_t y(x, v, t) = -(\partial_t^2 u)(x, -v, -t)$  and  $\nabla y(x, v, t) = \nabla \partial_t u(x, -v, -t)$ , and so

$$y \in H^1(-T, T; L^2(Q)) \cap L^2(-T, T; L^2(Q)), \quad \nabla y \in L^2(-T, T; L^2(Q)).$$

Moreover

$$\partial_t y + v \cdot \nabla y + \sigma_t y - \sigma_s \int_V p y dv' = f(x, v) \tilde{R}(x, v, t), \quad (x, v) \in Q, \quad -T < t < T. \quad (4.1)$$

By (1.10) we can choose  $\beta > 0$  such that  $\frac{\beta}{v_0^2} < 1$  and

$$T > \frac{\sup_{x \in \Omega} |x - x_0|}{\sqrt{\beta}}. \quad (4.2)$$

Later, we will bring  $\frac{\beta}{v_0^2}$  to 1 as close as possible.

Therefore we have

$$\varphi(x, \pm T) = |x - x_0|^2 - \beta T^2 < 0, \quad x \in \overline{\Omega}$$

and

$$\varphi(x, 0) > 0, \quad x \in \overline{\Omega}$$

by  $x_0 \notin \overline{\Omega}$ . Hence we can choose  $\delta > 0$  sufficiently small such that

$$\varphi(x, t) < -\delta, \quad -T < t < -T + 2\delta, \quad T - 2\delta < t < T, \quad x \in \overline{\Omega}$$

and

$$\varphi(x, t) > \delta, \quad -\delta < t < \delta, \quad x \in \overline{\Omega}.$$

We fix  $\chi \in C_0^\infty(\mathbb{R})$  such that  $0 \leq \chi \leq 1$  and

$$\chi(t) = \begin{cases} 1, & -T + 2\delta \leq t \leq T - 2\delta, \\ 0, & -T \leq t \leq -T + \delta, \quad T - \delta \leq t \leq T. \end{cases}$$

We set

$$z = \chi y, \quad w = \chi y e^{s\varphi}.$$

Then we have

$$\begin{aligned} & \partial_t z + v \cdot \nabla z + \sigma_t z - \sigma_s \int_V p(x, v, v', t) z(x, v', t) dv' \\ &= \chi(t) f(x, v) \tilde{R}(x, v, t) - (\partial_t \chi) y, \quad (x, v) \in Q, \quad -T < t < T, \\ & z(\cdot, \cdot, \pm T) = 0 \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} & \partial_t w + v \cdot \nabla w + \sigma_t w - \sigma_s \int_V p(x, v, v', t) w(x, v', t) dv' \\ &= \chi e^{s\varphi} f \tilde{R} - (\partial_t \chi) y e^{s\varphi} - s(\partial_t \varphi + (v \cdot \nabla \varphi)) w, \quad (x, v) \in Q, \quad -T < t < T, \\ & w(\cdot, \cdot, \pm T) = 0. \end{aligned} \tag{4.4}$$

Henceforth  $C > 0$  denotes generic constants which are independent of  $s > 0$  and  $f$ .

By  $\sigma_t \in L^\infty(Q \times (-T, T))$  and  $p \in L^\infty(\Omega \times V \times V \times (-T, T))$ , we see that

$$\begin{aligned} & \int_{-T}^T \int_Q \left| \sigma_s(x, v, t) \int_V p(x, v, v', t) z(x, v', t) dv' \right|^2 e^{2s\varphi(x, t)} dx dv dt \\ & \leq C \int_{-T}^T \int_Q \left( \int_V |\sigma_s(x, v, t)|^2 |p(x, v, v', t)|^2 |z(x, v', t)|^2 dv' \right) e^{2s\varphi(x, t)} dx dv dt \\ & \leq C \int_{-T}^T \int_\Omega \left( \int_V \left( \int_V |z(x, v', t)|^2 dv' \right) dv \right) e^{2s\varphi(x, t)} dx dt \leq C \int_{-T}^T \int_Q |z(x, v, t)|^2 e^{2s\varphi(x, t)} dx dv dt \end{aligned} \tag{4.5}$$

and

$$\int_{-T}^T \int_Q |\sigma_t(x, v, t) z(x, v, t)|^2 e^{2s\varphi(x, t)} dx dv dt \leq C \int_{-T}^T \int_Q |z(x, v, t)|^2 e^{2s\varphi(x, t)} dx dv dt.$$

Therefore, applying Lemma 3.1 (ii), we have

$$\begin{aligned} & s \int_{-T}^T \int_Q |z|^2 e^{2s\varphi(x, t)} dx dv dt \leq C \int_Q |f(x, v)|^2 \left( \int_{-T}^T e^{-2s\beta t^2} dt \right) e^{2s\varphi(x, 0)} dx dv \\ & + C \int_{-T}^T \int_Q |\partial_t \chi|^2 y^2 e^{2s\varphi(x, t)} dx dv dt + C e^{Cs} \int_{-T}^T \int_{\Gamma_+} (\nu \cdot v) y^2 dS dv dt \\ & + C \int_{-T}^T \int_Q |z|^2 e^{2s\varphi(x, t)} dx dv dt \end{aligned}$$

for all large  $s > 0$ . By absorbing the last term on the right-hand side into the left-hand side by choosing sufficiently large  $s > 0$ , and we obtain

$$\begin{aligned} s \int_{-T}^T \int_Q |z|^2 e^{2s\varphi(x,t)} dx dv dt &\leq C \int_Q |f(x,v)|^2 \left( \int_{-T}^T e^{-2s\beta t^2} dt \right) e^{2s\varphi(x,0)} dx dv \\ &\quad + C \int_{-T}^T \int_Q |\partial_t \chi|^2 y^2 e^{2s\varphi(x,t)} dx dv dt + C e^{Cs} \int_{-T}^T \int_{\Gamma_+} (\nu \cdot v) y^2 dS dv dt. \end{aligned} \quad (4.6)$$

Multiplying (4.4) by  $-2w(x, v, t)$  and integrating over  $Q \times (0, T)$ , we obtain

$$\begin{aligned} & - \int_0^T \int_Q \partial_t (w^2) dx dv dt - \int_0^T \int_Q v \cdot \nabla (w^2) dx dv dt \\ &= \int_0^T \int_Q 2\sigma_t w^2 dx dv dt - 2\sigma_s \int_0^T \int_Q \left( \int_V p w(x, v', t) w(x, v, t) dv' \right) dx dv dt \\ & - \int_0^T \int_Q 2(\chi e^{s\varphi} f \tilde{R} w - (\partial_t \chi) y e^{s\varphi} w - s(\partial_t \varphi + v \cdot \nabla \varphi) w^2) dx dv dt. \end{aligned}$$

By  $w = \chi(\partial_t u) e^{s\varphi}$  for  $t > 0$ , we have  $w = 0$  on  $\Gamma_- \times (-T, T)$ . Therefore we have

$$\begin{aligned} & [\text{the left-hand side}] \\ &= \int_Q w(x, v, 0)^2 dx dv - \int_0^T \int_{\partial\Omega} \int_V w^2 (\nu \cdot v) dS dv dt \\ &= \int_Q w(x, v, 0)^2 dx dv - \int_0^T \left( \int_{\Gamma_+} + \int_{\Gamma_-} \right) w^2 (\nu \cdot v) dS dv dt \\ &\geq \int_Q w(x, v, 0)^2 dx dv - \int_0^T \int_{\Gamma_+} w^2 (\nu \cdot v) dS dv dt \\ &= \int_Q |y(x, v, 0)|^2 e^{2s\varphi(x,0)} dx dv - \int_0^T \int_{\Gamma_+} \chi^2 |\partial_t u|^2 (\nu \cdot v) e^{2s\varphi(x,t)} dS dv dt. \end{aligned}$$

By the Cauchy-Schwarz inequality and an argument similar to (4.5), we have

$$\begin{aligned} & | [\text{the right-hand side}] | \\ &\leq C s \int_{-T}^T \int_Q z^2 e^{2s\varphi(x,t)} dx dv dt + C \int_{-T}^T \int_Q z^2 e^{2s\varphi(x,t)} dx dv dt \\ &+ C \int_{-T}^T \int_Q (\partial_t \chi)^2 y^2 e^{2s\varphi(x,t)} dx dv dt + C \int_{-T}^T \int_Q f^2 e^{2s\varphi(x,t)} dx dv dt. \end{aligned}$$

Therefore

$$\int_Q |\partial_t u(x, v, 0)|^2 e^{2s\varphi(x,0)} dx dv \leq C \int_{-T}^T \int_Q |f|^2 e^{2s\varphi(x,t)} dx dv dt + C \int_{-T}^T \int_Q (\partial_t \chi)^2 y^2 e^{2s\varphi(x,t)} dx dv dt$$

$$+Cs \int_{-T}^T \int_Q z^2 e^{2s\varphi(x,t)} dx dv dt + \int_{-T}^T \int_{\Gamma_+} |\partial_t u|^2 (\nu \cdot v) e^{2s\varphi(x,t)} dS dv dt. \quad (4.7)$$

Substituting (4.6) into the third term on the right-hand side of (4.7), we obtain

$$\begin{aligned} & \int_Q |\partial_t u(x, v, 0)|^2 e^{2s\varphi(x,0)} dx dv \\ \leq & C \int_Q |f(x, v)|^2 \left( \int_{-T}^T e^{-2s\beta t^2} dt \right) e^{2s\varphi(x,0)} dx dv + C \int_{-T}^T \int_Q |\partial_t \chi|^2 y^2 e^{2s\varphi(x,t)} dx dv dt \\ + & C e^{Cs} \int_{-T}^T \int_{\Gamma_+} (\nu \cdot v) y^2 dS dv dt. \end{aligned}$$

Here by the definition of  $\chi$ , we see that  $\partial_t \chi \neq 0$  only if  $-T + \delta \leq t \leq -T + 2\delta$  or  $T - 2\delta \leq t \leq T - \delta$ , which implies  $\varphi(x, t) < -\delta$ . Hence

$$\int_{-T}^T \int_Q |\partial_t \chi|^2 y^2 e^{2s\varphi(x,t)} dx dv dt \leq C e^{-2s\delta} \int_{-T}^T \int_Q y^2 dx dv dt.$$

By the Lebesgue theorem, we have

$$\int_{-T}^T e^{-2s\beta t^2} dt = o(1) \quad \text{as } s \rightarrow \infty.$$

By  $u(\cdot, \cdot, 0) = 0$ , we obtain  $\partial_t u(x, v, 0) = f(x, v)R(x, v, 0)$ . Consequently,  $R(x, v, 0) \neq 0$  for  $(x, v) \in \overline{Q}$ , we obtain

$$\int_Q |f(x, v)|^2 e^{2s\varphi(x,0)} dx dv \quad (4.8)$$

$$\begin{aligned} \leq & o(1) \int_Q |f(x, v)|^2 e^{2s\varphi(x,0)} dx dv + C e^{-2s\delta} \int_{-T}^T \int_Q y^2 dx dv dt \\ + & C e^{Cs} \int_{-T}^T \int_{\Gamma_+} (\nu \cdot v) y^2 dS dv dt \end{aligned}$$

for all large  $s > 0$ . By (2.3), noting that  $y$  is the extension of  $\partial_t u$  to  $t < 0$ , we have

$$\int_Q |y(x, v, t)|^2 dx dv \leq C \|f\|_{L^2(Q)}, \quad -T \leq t \leq T.$$

Note that  $|x - x_0| \geq \varepsilon > 0$ ,  $x \in \overline{\Omega}$  with some constant  $\varepsilon > 0$  because  $x_0 \notin \overline{\Omega}$ . By (4.8) we obtain

$$\begin{aligned} & (1 - o(1)) e^{2s\varepsilon} \int_Q |f(x, v)|^2 dx dv \leq (1 - o(1)) \int_Q |f(x, v)|^2 e^{2s\varphi(x,0)} dx dv \\ \leq & C e^{-2s\delta} T \int_Q |f(x, v)|^2 dx dv + C e^{Cs} \int_{-T}^T \int_{\Gamma_+} (\nu \cdot v) y^2 dS dv dt \end{aligned}$$

for all large  $s > 0$ . Since  $y$  is the extension of  $\partial_t u$  to  $t < 0$ , we have

$$\{(1 - o(1))e^{2s\varepsilon} - Ce^{-2s\delta}T\} \int_Q |f(x, v)|^2 dx dv \leq Ce^{Cs} \int_{\Gamma_+} \int_0^T (\nu \cdot v) |\partial_t u|^2 dS dv dt$$

for all large  $s > 0$ . Choosing sufficiently large  $s > 0$ , we complete the proof.

## References

- [1] Arridge S R 1999 Optical tomography in medical imaging *Inverse Problems* **15** R41–R93
- [2] Arridge S R and Schotland J C 2009 Optical tomography: forward and inverse problems *Inverse Problems* **25** 123010
- [3] Bal G 2009 Inverse transport theory and applications *Inverse Problems* **25** 053001
- [4] Bal G and Jollivet A 2009 Time-dependent angularly averaged inverse transport *Inverse Problems* **25** 075010
- [5] Bal G and Jollivet A 2010 Stability for time-dependent inverse transport *SIAM J. Math. Anal.* **42** 679–700
- [6] Bukhgeim A L and Klibanov M V 1981 Global uniqueness of a class of multidimensional inverse problems *Soviet Math. Dokl.* **24** 244–247
- [7] Carleman T 1939 Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendentes *Ark. Mat. Astr. Fys.* **2B** 1–9
- [8] Case K M and Zweifel P F 1967 *Linear Transport Theory* (Boston: Addison-Wesley)
- [9] Chandrasekhar S 1960 *Radiative Transfer* (New York: Dover Publications)
- [10] Choulli M and Stefanov P 1996 Inverse scattering and inverse boundary value problems for the linear Boltzmann equation *Comm. P. D. E.* **21** 763–85
- [11] Davison B and Sykes J B 1957 *Neutron Transport Theory* (Oxford: Oxford University Press)
- [12] Franceschini M A, Joseph D K, Huppert T J, Diamond S G and Boas D A 2006 Diffuse optical imaging of the whole head *J. Biomed. Opt.* **11** 054007

- [13] Hörmander L 1963 *Linear Partial Differential Operators* (Berlin: Springer-Verlag)
- [14] Huppert T J, Diamond S G, Franceschini M A and Boas D A 2009 HomER: a review of time-series analysis methods for near-infrared spectroscopy of the brain *Appl. Opt.* **48** D280–D298
- [15] Imanuvilov O and Yamamoto M 2001 Global uniqueness and stability in determining coefficients of wave equations *Comm. P. D. E.* **26** 1409–25
- [16] Imanuvilov O and Yamamoto M 2001 Global Lipschitz stability in an inverse hyperbolic problem by interior observations *Inverse Problems* **17** 717–28
- [17] Isakov V 1990 *Inverse Source Problems* (Providence, Rhode Island: American Mathematical Society)
- [18] Isakov V 1993 Carleman type estimates in an anisotropic case and applications, *J. Differential Equations* **105** 217–38
- [19] Isakov V 2006 *Inverse Problems for Partial Differential Equations* (Berlin: Springer-Verlag)
- [20] Klibanov M V 1984 Inverse problems in the glargeh and Carleman boun *Diff. Eq.* **20** 755–60
- [21] Klibanov M V 1992 Inverse problems and Carleman estimates *Inverse Problems* **8** 575–96
- [22] Klibanov M V and Pamyatnykh S E 2006 Lipschitz stability of a non-standard problem for the non-stationary transport equation via a Carleman estimate *Inverse Problems* **22** 881–90
- [23] Klibanov M V and Pamyatnykh S E 2008 Global uniqueness for a coefficient inverse problem for the non-stationary transport equation via Carleman estimate *J. Math. Anal. Appl.* **343** 352–65
- [24] Klibanov M V and Timonov A 2004 *Carleman Estimates for Coefficient Inverse Problems and Numerical Applications* (Utrecht: VSP)
- [25] Klibanov M V and Yamamoto M 2007 Exact controllability for the time dependent transport equation *SIAM J. Control Optim.* **46** 2071–195

- [26] Lavrent'ev M M, Romanov V G and Shishat'skii S P 1986 *Ill-posed Problems of Mathematical Physics and Analysis* (Providence, Rhode Island: American Mathematical Society)
- [27] McDowall S, Stefanov P and Tamasan A 2010 Stability of the gauge equivalent classes in inverse stationary transport *Inverse Problems* **26** 025006
- [28] Prilepko A I and Ivankov A L 1984 Inverse problems for the time-dependent transport equation *Soviet Math. Dokl.* **29** 559–64
- [29] Sobolev V V 1975 *Light Scattering in Planetary Atmospheres* (Oxford: Pergamon Press)
- [30] Stefanov P 2003 *Inverse problems in transport theory, Inside Out: Inverse Problems and Applications* ed. G.Uhlmann (Cambridge: Cambridge University Press) pp 111–31
- [31] Stefanov P and Tamasan A 2009 Uniqueness and non-uniqueness in inverse radiative transfer *Proc. Amer. Math. Soc.* **137** 2335–44
- [32] Yamamoto M 2009 Carleman estimates for parabolic equations and applications *Inverse Problems* **25** 123013 (75pp)